

SIGNATURES OF LEFSCHETZ FIBRATIONS

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ABSTRACT. Let M be a smooth 4-manifold which admits a genus g C^∞ -Lefschetz fibration over D^2 or S^2 . We develop a technique to compute the signature of M using the global monodromy of this fibration. As a corollary we prove that there is no hyperelliptic Lefschetz fibration (of any genus) over S^2 with only reducible singular fibers. We also prove that for each $g \geq 1$, there exist

$$k_g \leq 10 - \frac{6g+4}{g^2} \text{ and } l_g \leq 2g - 10 + \frac{4g+4}{g^2}$$

such that if a 4-manifold admits a hyperelliptic C^∞ -Lefschetz fibration of genus g over S^2 then its signature σ and its Euler characteristic e satisfy the inequality $c_1^2 \leq k_g \chi + l_g$, where $c_1^2 = 3\sigma + 2e$ and $\chi = \frac{1}{4}(\sigma + e)$. In particular, we show that we can choose $k_2 = 6$, $l_2 = -3$, $k_3 = 7.25$, $l_3 = -2.75$, $k_4 = 8.25$ and $l_4 = -0.75$.

0. INTRODUCTION

The signature of a smooth 4-manifold which admits a hyperelliptic Lefschetz fibration of genus g over a closed surface can be computed using the *local signature formula*, given by Matsumoto ([M1], [M2]) for $g = 1, 2$ and more recently extended by Endo [E] for $g \geq 3$.

In this paper we present a method to compute the signature of a smooth 4-manifold which admits an arbitrary (not necessarily hyperelliptic) Lefschetz fibration of any genus over D^2 or S^2 . A Lefschetz fibration on a smooth 4-manifold M gives rise to a handlebody description of M , which is determined by a sequence of *vanishing cycles*. We use this handlebody description [K] and *Wall's nonadditivity formula* for signatures [W] in order to compute the signature of M . Hence we calculate a ‘*relative signature*’ corresponding to each singular fiber of the given fibration on M .

Despite the fact that the vanishing cycles are defined up to isotopy, our technique shows that the signature of a 4-manifold which admits a Lefschetz fibration depends only on the algebraic data given by the homology classes of the vanishing cycles.

Recent results in symplectic topology shows that Lefschetz fibrations provide a topological characterization of symplectic 4-manifolds: Donaldson [D] has shown that, after perhaps blowing up, a closed symplectic 4-manifold admits a Lefschetz fibration over S^2 , and conversely Gompf [GS] has

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shown that most Lefschetz fibrations are symplectic – the exceptions all have fiber-genus 1 and are blow-ups of torus fibrations with no critical points. Hence by computing the signatures of Lefschetz fibrations of any genus we hope to answer some of the problems in the *geography* of symplectic 4-manifolds ([S1], [S2], [GS]).

It is well-known that the complex surfaces satisfy the BMY-inequality $c_1^2 \leq 9\chi$. In [S2], Stipsicz proves that $0 \leq c_1^2 \leq 10\chi$ for the (relatively minimal) genus $g(\geq 1)$ Lefschetz fibrations over closed surfaces of nonzero genus. His result, however, does not extend to cover the fibrations over S^2 . Hence the symplectic version of the BMY-inequality is still missing. Stipsicz [S2] also points out that $c_1^2 \leq 10\chi + 2g - 1$ holds for any Lefschetz fibration.

Lefschetz fibrations on smooth 4-manifolds are generalized from holomorphic Lefschetz fibrations on complex surfaces. It is conjectured by G.Tian that any irreducible hyperelliptic (smooth) Lefschetz fibration is holomorphic. Here irreducible means that the global monodromy of the fibration does not contain a Dehn twist about a separating curve. There are, however, noncomplex smooth 4-manifolds which admit reducible hyperelliptic Lefschetz fibrations ([FS], [OS]).

We note that the signature of an irreducible hyperelliptic Lefschetz fibration is always negative.

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1. PRELIMINARIES

1.1. Mapping Class Groups.

Definition . Let Σ_g be a closed oriented surface of genus g . Let $Diff^+(\Sigma_g)$ be the group of all orientation preserving self diffeomorphisms of Σ_g . Let $Diff_0^+(\Sigma_g)$ be the subgroup of $Diff^+(\Sigma_g)$ consisting of all self diffeomorphisms isotopic to the identity. Then we define the *mapping class group* of genus g as

$$\mathcal{M}_g = Diff^+(\Sigma_g)/Diff_0^+(\Sigma_g).$$

The *hyperelliptic mapping class group* \mathcal{H}_g of genus g is defined ([E]) as the subgroup of \mathcal{M}_g which consists of all isotopy classes commuting with the isotopy class of the hyperelliptic involution $\iota : \Sigma_g \longrightarrow \Sigma_g$.

It is known ([BH]) that the hyperelliptic mapping class group \mathcal{H}_g agrees with the mapping class group \mathcal{M}_g for $g = 1, 2$.

We will denote a positive Dehn twist about a simple closed curve γ by $D(\gamma) \in \mathcal{M}_g$ and we will use the functional notation for the products in \mathcal{M}_g , e.g., $D(\beta)D(\alpha)$ will denote the composition where we apply $D(\alpha)$ first and then $D(\beta)$.

1.2. Smooth Lefschetz Fibrations.

Definition . Let M be a compact, oriented smooth 4-manifold, and let B be a compact, oriented 2-manifold. A proper smooth map $f : M \rightarrow B$ is a (C^∞ -) *Lefschetz fibration* if there exist points $b_1, \dots, b_m \in \text{interior}(B)$ such that

- (1) $\{b_1, \dots, b_m\}$ are the critical values of f , with $p_i \in f^{-1}(b_i)$ a unique critical point of f , for each i , and
- (2) about each b_i and p_i , there are complex coordinate neighborhoods such that locally f can be expressed as $f(z_1, z_2) = z_1^2 + z_2^2$.

It is a consequence of this definition that

$$f|_{f^{-1}(B - \{b_1, \dots, b_m\})} : f^{-1}(B - \{b_1, \dots, b_m\}) \rightarrow B - \{b_1, \dots, b_m\}$$

is a smooth fiber bundle over $B - \{b_1, \dots, b_m\}$ with fiber diffeomorphic to a 2-manifold Σ_g , and so we refer to f (and sometimes also the manifold M) as a *genus g Lefschetz fibration* (or a *Lefschetz fibration of genus g*). Two genus g Lefschetz fibrations $f : M \rightarrow B$ and $f' : M' \rightarrow B'$ are *equivalent* if there are diffeomorphisms $\Phi : M \rightarrow M'$ and $\phi : B \rightarrow B'$ such that $f'\Phi = \phi f$.

We always assume that our Lefschetz fibrations are *relatively minimal*, namely that no fiber contains an embedded 2-sphere of self-intersection number -1 . We also assume that there is at least one singular fiber in each fibration.

If $f : M \rightarrow D^2$ is a smooth genus g Lefschetz fibration, then we can use the Lefschetz fibration to produce a handlebody description of M . We select a regular value $b_0 \in \text{interior}(D^2)$ of f , an identification $f^{-1}(b_0) \cong \Sigma_g$, and a collection of arcs s_i in $\text{interior}(D^2)$ with each s_i connecting b_0 to b_i , and otherwise disjoint from the other arcs. We also assume that the critical values are indexed so that the arcs s_1, \dots, s_m appear in order as we travel counterclockwise in a small circle about b_0 . Let V_0, \dots, V_m denote a collection of small disjoint open disks with $b_i \in V_i$ for each i .

To build our description of M , we observe first that $f^{-1}(V_0) \cong \Sigma_g \times D^2$, with $\partial V_0 \cong \Sigma_g \times S^1$. Enlarging V_0 to include the critical value b_1 , it can be shown that $f^{-1}(V_0 \cup \nu(s_1) \cup V_1)$ is diffeomorphic to $\Sigma_g \times D^2$ with a 2-handle h_1 attached along a circle γ_1 contained in a fiber $\Sigma_g \times \text{pt.} \subset \Sigma_g \times S^1$.

Moreover, condition (2) in the definition of a Lefschetz fibration requires that h_1 is attached with a framing -1 relative to the natural framing on γ_1 inherited from the product structure of ∂V_0 . γ_1 is called a vanishing cycle. In addition, $\partial((\Sigma_g \times D^2) \cup h_1)$ is diffeomorphic to a Σ_g -bundle over S^1 whose monodromy is given by $D(\gamma_1)$, a positive Dehn twist about γ_1 . Continuing counterclockwise about b_0 , we add the remaining critical values to our description, yielding that

$$M_0 \cong f^{-1}(V_0 \cup (\bigcup_{i=1}^m \nu(s_i)) \cup (\bigcup_{i=1}^m V_i))$$

is diffeomorphic to $(\Sigma_g \times D^2) \cup (\bigcup_{i=1}^m h_i)$, where each h_i is a 2-handle attached along a vanishing cycle γ_i in a Σ_g -fiber in $\Sigma_g \times S^1$ with relative framing -1 . Furthermore,

$$\partial M_0 \cong \partial((\Sigma_g \times D^2) \cup (\bigcup_{i=1}^m h_i))$$

is a Σ_g -bundle over S^1 with monodromy given by the composition $D(\gamma_m) \cdots D(\gamma_1)$. We will refer to the cyclically ordered collection $(D(\gamma_1), \dots, D(\gamma_m))$ (or the product $D(\gamma_m) \cdots D(\gamma_1)$) as the *global monodromy* of this fibration.

We can extend this description to Lefschetz fibrations over S^2 as follows:

Assume that $f : M \rightarrow S^2$ is a smooth genus g Lefschetz fibration. Let $M_0 = M - \nu(f^{-1}(b))$, where $\nu(f^{-1}(b)) \cong \Sigma_g \times D^2$ denotes a regular neighborhood of a nonsingular fiber $f^{-1}(b)$. Then $f|_{M_0} : M_0 \rightarrow D^2$ is a smooth Lefschetz fibration. If $(D(\gamma_1), \dots, D(\gamma_m))$ is the global monodromy of the fibration $f|_{M_0} : M_0 \rightarrow D^2$, then $D(\gamma_m) \cdots D(\gamma_1)$ is isotopic to the identity since also $\partial M_0 \cong \Sigma_g \times S^1$. Finally, to extend our description of M_0 to M , we reattach $\Sigma_g \times D^2$ to $(\Sigma_g \times D^2) \cup (\bigcup_{i=1}^m h_i)$ via a Σ_g -fiber preserving map of the boundary. This extension is unique up to equivalence for $g \geq 2$ [K].

Remark . Although the description of the monodromy corresponding to each individual critical value b_i as a Dehn twist depends on the choice of arc s_i , other choices of arcs (and of the central identification $f^{-1}(x_0) \cong \Sigma_g$) do not change the Lefschetz fibration on M_0 , up to equivalence [GS].

Definition . Let $f : M \rightarrow S^2$ be a smooth genus g Lefschetz fibration with global monodromy $(D(\gamma_1), \dots, D(\gamma_m))$. We will call $f : M \rightarrow S^2$ a *hyperelliptic Lefschetz fibration* of genus g iff there exists $h \in \mathcal{M}_g$ such that $hD(\gamma_i)h^{-1} \in \mathcal{H}_g$ for all i , $1 \leq i \leq m$.

Remark . All Lefschetz fibrations of genus one and genus two are hyperelliptic since $\mathcal{H}_g = \mathcal{M}_g$ for $g = 1, 2$.

1.3. Wall's Non-Additivity Formula.

If two compact oriented 4-manifolds are glued by an orientation reversing diffeomorphism of their boundaries, then the signature of their union is the sum of their signatures. This is known as the Novikov additivity. But it is often desirable to consider the more general case of gluing: along a common submanifold, which may itself have boundary, of the boundaries of the original manifolds. However, the Novikov additivity does not hold in this general case. Wall [W] derives a formula for the deviation from additivity in the general case, which is known as the Wall's nonadditivity formula.

We will give a specific case of his formula:

Let X_- , X_0 , X_+ be 3-manifolds and Y_- and Y_+ be 4-manifolds such that

$$\partial X_- = \partial X_0 = \partial X_+ = Z,$$

$$\partial Y_- = X_- \cup X_0,$$

$$\partial Y_+ = X_0 \cup X_+;$$

write $Y = Y_- \cup Y_+$ and $X = X_- \cup X_0 \cup X_+$. (Figure 1)

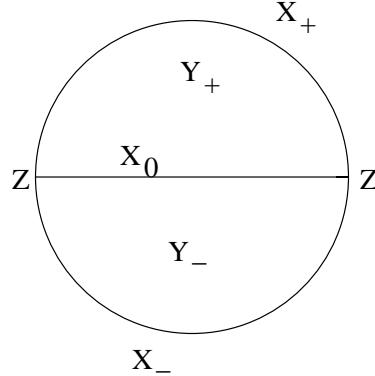


FIGURE 1.

Suppose that Y is oriented inducing orientations of Y_- and Y_+ . Orient the rest so that

$$\partial_*[Y_-] = [X_0] - [X_-],$$

$$\partial_*[Y_+] = [X_+] - [X_0],$$

$$\partial_*[X_-] = \partial_*[X_0] = \partial_*[X_+] = [Z].$$

Write $V = H_1(Z; \mathbb{R})$; let A, B, C be the kernels of the maps induced by the inclusions of Z in X_-, X_0 and X_+ respectively. Then $\dim A = \dim B = \dim C = \frac{(\dim V)}{2}$.

Let Φ denote the oriented intersection numbers in Z . Let $W = \frac{C \cap (A+B)}{(C \cap A) + (C \cap B)}$. Then Wall defines a symmetric bilinear map $\Psi : W \times W \rightarrow \mathbb{R}$ as follows: First define $\Psi' : C \cap (A+B) \times C \cap (A+B) \rightarrow \mathbb{R}$

by

$$\Psi'(c, c') = \Phi(c, a')$$

where $a' + b' + c' = 0$ for some $b' \in B$. Then Ψ' induces a map Ψ on W .

The signature of the symmetric bilinear map Ψ will be denoted by $\sigma(V; C, A, B)$.

We also denote the signature of a 4-manifold M as $\sigma(M)$ in the rest of this paper.

We are now ready to state Wall's formula:

Theorem 1. [W] $\sigma(Y) = \sigma(Y_-) + \sigma(Y_+) - \sigma(V; C, A, B)$.

1.4. Local Signature Formula.

The following theorem was proven by Matsumoto for $g = 1, 2$ using the fact that the cohomology class of Meyer's signature cocycle has finite order in the cohomology group $H^2(\mathcal{M}_g, \mathbb{Z})$. Recently, Endo proved the $g \geq 3$ case by observing the finiteness of the order of the cohomology class of the signature cocycle restricted to the hyperelliptic mapping class group \mathcal{H}_g .

Theorem 2. ([M1], [M2], [E]) *Let M be a 4-manifold which admits a hyperelliptic Lefschetz fibration of genus g over S^2 . Let n and $s = \sum_{h=1}^{[\frac{g}{2}]} s_h$ be the numbers of nonseparating and separating vanishing cycles in the global monodromy of this fibration, respectively. Then*

$$\sigma(M) = -\frac{g+1}{2g+1}n + \sum_{h=1}^{[\frac{g}{2}]} \left(\frac{4h(g-h)}{2g+1} - 1 \right) s_h.$$

Remark . Here s_h denotes the number of separating vanishing cycles which separate the genus g surface into two surfaces one of which has genus h .

2. MAIN THEOREMS

In this section we explain our main idea and establish the main theorems to develop an algorithm to compute the signature of a 4-manifold which admits a Lefschetz fibration over D^2 or S^2 using the global monodromy of this fibration.

Definition . Let X be a 4-manifold with boundary $\partial X \cong \Sigma_g \times I/(x, 1) \sim (\phi(x), 0)$, where ϕ is a self-diffeomorphism of Σ_g . Let X' denote the resulting 4-manifold after attaching a 2-handle to X along a simple closed curve γ on $\Sigma_g \times \{pt\}$ with framing -1 (relative to the product framing). Then $\sigma(\phi, \gamma)$ is defined as $\sigma(X') - \sigma(X)$.

Theorem 3. *Let M be a 4-manifold which admits a genus g Lefschetz fibration over D^2 or S^2 . Let $(D(\gamma_1), \dots, D(\gamma_t))$ be the global monodromy of this fibration. Let $D(\gamma_0)$ denote the identity map. Then*

$$\sigma(M) = \sum_{i=1}^t \sigma(D(\gamma_{i-1}) \cdots D(\gamma_0), \gamma_i),$$

where $\sigma(D(\gamma_{i-1}) \cdots D(\gamma_1), \gamma_i) \in \{-1, 0, +1\}$ for all i , $1 \leq i \leq t$.

Proof. It suffices to prove the result for Lefschetz fibrations over D^2 . (By Novikov additivity it extends to Lefschetz fibrations over S^2 .) We use the handlebody description of M and Wall's formula as follows:

We start with a copy of $M_0 = \Sigma_g \times D^2$. We attach a 2-handle to M_0 along γ_1 with framing -1 . Let M_1 denote the resulting manifold. Then ∂M_1 will have monodromy $D(\gamma_1)$, a positive Dehn twist about γ_1 . Now we attach another 2-handle to M_1 along γ_2 . Let M_2 denote the resulting manifold. Proceeding in this manner we get the manifolds M_1, M_2, \dots, M_t .

We are going to apply Wall's formula at each step of this construction to compute the signature of M . In order to apply Wall's formula we set up the following notation:

ϕ : a self-diffeomorphism of Σ_g

X : a 4-manifold with boundary $\partial X \cong \Sigma_g \times I/(x, 1) \sim (\phi(x), 0)$.

γ : a simple closed curve embedded in a fiber $\Sigma_g \times \{pt\}$.

X' : resulting 4-manifold after attaching a 2-handle to X along a simple closed curve γ on $\Sigma_g \times \{pt\}$ with framing -1 (relative to the product framing).

$\nu(\gamma)$: a regular neighborhood of γ in ∂X .

i_* : the induced map on the homology by the inclusion of appropriate spaces.

Now we define $Y_+, Y_-, X_+, X_0, X_-, Z$ in Wall's formula as follows:

$Y_- = D^2 \times D^2$, $Y_+ = X$,

$\partial Y_- = \partial(D^2 \times D^2) = S^1 \times D^2 \cup D^2 \times S^1$,

$\partial Y_+ = \Sigma_g \times I/(x, 1) \sim (\phi(x), 0)$,

$X_0 = S^1 \times D^2 \cong \nu(\gamma)$, $X_- = D^2 \times S^1$, $X_+ = \partial X - \overset{\circ}{\nu}(\gamma)$,

$Z = S^1 \times S^1 \cong \partial \nu(\gamma) \cong \partial(\partial X - \overset{\circ}{\nu}(\gamma))$.

Hence,

$$A = \text{Ker}(i_* : H_1(S^1 \times S^1; \mathbb{R}) \rightarrow H_1(D^2 \times S^1; \mathbb{R})),$$

$$B = \text{Ker}(i_* : H_1(S^1 \times S^1; \mathbb{R}) \rightarrow H_1(S^1 \times D^2; \mathbb{R})),$$

$$C = \text{Ker}(i_* : H_1(\partial \nu(\gamma); \mathbb{R}) \rightarrow H_1(\partial X - \overset{\circ}{\nu}(\gamma); \mathbb{R})).$$

Let l be the longitude $S^1 \times \{pt\}$ and m be the meridian $\{pt\} \times \partial D^2$ of $X_0 = S^1 \times D^2$. Then $A = \langle [l] \rangle$ and $B = \langle [m] \rangle$. We also know that C is a 1-dimensional subspace of

$$H_1(S^1 \times S^1; \mathbb{R}) = \langle [l], [m] \rangle \cong \mathbb{R}^2.$$

Let Φ be the intersection form on $Z = S^1 \times S^1$ and $W = \frac{C \cap (A+B)}{(C \cap A) + (C \cap B)} = \frac{C}{(C \cap A) + (C \cap B)}$. Hence $W = \{0\}$ if $C = A$ or $C = B$ and $W = C$ otherwise. Now assume that $C \neq A$ and $C \neq B$. Then $C = \langle c \rangle = \langle p[l] + q[m] \rangle$ for some $p, q \in \mathbb{R}$ and $\Psi(c, c) = \Phi(c, a')$ where $c + a' + b' = 0$ for some $a' \in A$ and $b' \in B$. (Ψ is the bilinear form in Wall's formula). Let $a' = -p[l]$ and $b' = -q[m]$. Then we have,

$$\Psi(c, c) = \Phi(c, -p[l]) = \Phi(p[l] + q[m], -p[l]) = -pq\Phi([m], [l]) = pq.$$

Therefore signature of Ψ is given by the sign of pq .

Hence by Wall's formula

$$\begin{aligned} \sigma(X') &= \sigma(X) + \sigma(D^2 \times D^2) - \sigma(\mathbb{R}^2; C, A, B) \\ &= \sigma(X) - \text{signature}(\Psi) = \sigma(X) - \text{sign}(pq). \end{aligned}$$

This proves the theorem by setting $X = M_i$ for $i = 1, 2, \dots, t-1$.

□

So the idea to compute the signature of a genus g Lefschetz fibration is very simple. For each 2-handle that we attach to $\Sigma_g \times D^2$ along a vanishing cycle, there is a corresponding *relative signature* $\in \{-1, 0, +1\}$. Once we attach all the 2-handles, the sum of the relative signatures will be signature of the 4-manifold. The difficulty is to compute the relative signatures using the vanishing cycles (or more precisely using only the homology classes of the vanishing cycles). The following technical theorems will be helpful in computations.

Theorem 4. *In addition to the notation above, let $\{a_1, b_1, a_2, b_2, \dots, a_g, b_g\}$ be the standard basis for $H_1(\Sigma_g; \mathbb{R})$. (We will use the letters a_i and b_i also to denote the curves which represent the homology classes a_i and b_i , respectively, for $1 \leq i \leq g$.) Then*

(1) *If γ is a nonseparating curve, then there exists a longitude l' and a meridian m' of $\partial(\partial X - \overset{\circ}{\nu}(\gamma))$ such that*

$$\begin{aligned} i_*[l'] &= [\gamma] \in H_1(\partial X - \overset{\circ}{\nu}(\gamma); \mathbb{R}) \\ i_*[m'] &= \frac{e - \phi_*(e)}{e \cdot [\gamma]} \in H_1(\partial X - \overset{\circ}{\nu}(\gamma); \mathbb{R}) \end{aligned}$$

for all $e \in \{a_1, b_1, a_2, b_2, \dots, a_g, b_g\}$, where $e \cdot [\gamma] \neq 0$.

(2) *If γ is a separating curve, then $\sigma(X') = \sigma(X) - 1$, i.e., $\sigma(\phi, \gamma) = -1$.*

Proof. We recall that ∂X is a mapping torus, i.e., $\partial X \cong \Sigma_g \times I/(x, 1) \sim (\phi(x), 0)$ and γ is a curve on a fiber $\Sigma_g \times \{pt\}$. We note that a regular neighborhood of γ in Σ_g is given by $\gamma \times I_1$. Hence a regular neighborhood of γ in ∂X is given by $\gamma \times I_1 \times I_2$ where I_2 is a small neighborhood of the $\{pt\}$ in $S^1 = I/(1 \sim 0)$. This neighborhood of γ is called the product neighborhood [K].

Now let us push off γ to the boundary of $\partial X - \overset{\circ}{\nu}(\gamma)$. Denote the push off of γ as l' . Moreover if we identify $I_1 \times I_2$ as D^2 and denote ∂D^2 as m' , then $\{l', m'\}$ will be a longitude-meridian pair for $\partial(\partial X - \overset{\circ}{\nu}(\gamma))$. Then clearly

$$i_*[l'] = [\gamma] \in H_1(\partial X - \overset{\circ}{\nu}(\gamma); \mathbb{R})$$

On the other hand, to find the image of m' we observe the following:

Assume that $e.[\gamma] = 1$ for some $e \in \{a_1, b_1, a_2, b_2, \dots, a_g, b_g\}$. Then we locally have the picture in Figure 2 in a neighborhood of the point where e and γ meet.

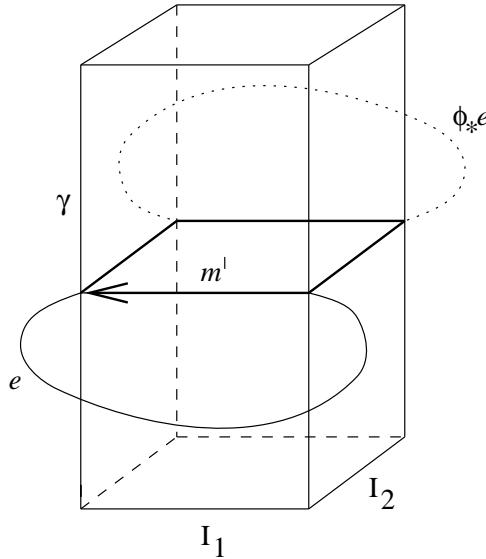


FIGURE 2.

This proves that

$$i_*[m'] = e - \phi_*(e) \in H_1(\partial X - \overset{\circ}{\nu}(\gamma); \mathbb{R}).$$

Now assume that $e.[\gamma] = -1$ for some $e \in \{a_1, b_1, a_2, b_2, \dots, a_g, b_g\}$. Then we locally have a similar picture in a neighborhood of the point where e and γ meet, except for the orientations.

This proves that

$$i_*[m'] = \phi_*(e) - e \in H_1(\partial X - \overset{\circ}{\nu}(\gamma); \mathbb{R}).$$

Since these are local results it follows combining these two observations that

$$i_*[m'] = \frac{e - \phi_*(e)}{e.[\gamma]} \in H_1(\partial X - \overset{\circ}{\nu}(\gamma); \mathbb{R}).$$

To prove the second part of the theorem we note that if γ is a separating curve in Σ_g then it is homologically trivial. Thus $i_*[l'] = 0$. This implies that $\text{Ker}(i_*) = \langle [l'] \rangle$.

Note that, in terms of the bases $\{[l], [m]\}$ of $H_1(\partial(S^1 \times D^2); \mathbb{R}) = H_1(S^1 \times S^1; \mathbb{R})$ and $\{[l'], [m']\}$ of $H_1(\partial(\partial X - \overset{\circ}{\nu}(\gamma)); \mathbb{R})$, attaching a 2-handle by -1 framing means that we identify $[l]$ with $[l'] - [m']$ and $[m]$ with $[m']$.

So if we transform the $\text{Ker}(i_*)$ to the $\{[l], [m]\}$ plane we see that $\text{Ker}(i_*) = C = W = \langle [l] + [m] \rangle$ which implies that $\sigma(X') = \sigma(X) - (+1)$ (cf. Theorem 3).

□

Proposition 5. *We use the same notation as in Theorem 4.*

- (1) *Let $\gamma = a_i$ for some i , $1 \leq i \leq g$. Then $H_1(\partial X - \overset{\circ}{\nu}(a_i); \mathbb{R}) = \langle a_1, b_1, a_2, b_2, \dots, a_g, b_g, b'_i, t \mid a_j = \phi_* a_j \text{ for all } j, b_j = \phi_* b_j \text{ for all } j \neq i, b'_i = \phi_* b_i \rangle$. Moreover $i_*[l'] = a_i$ and $i_*[m'] = b_i - b'_i$.*
- (2) *Let $\gamma = b_i$ for some i , $1 \leq i \leq g$. Then $H_1(\partial X - \overset{\circ}{\nu}(b_i); \mathbb{R}) = \langle a_1, b_1, a_2, b_2, \dots, a_g, b_g, a'_i, t \mid b_j = \phi_* b_j \text{ for all } j, a_j = \phi_* a_j \text{ for all } j \neq i, a'_i = \phi_* a_i \rangle$. Moreover $i_*[l'] = b_i$ and $i_*[m'] = a'_i - a_i$.*

Proof. Assume that $\gamma = a_i$ for some i , $1 \leq i \leq g$. We first use Van-Kampen's theorem to compute $\pi_1(\partial X - \overset{\circ}{\nu}(a_i))$. Write $\partial X - \overset{\circ}{\nu}(a_i) = E_1 \cup E_2$ as follows: Let $E_1 = \Sigma_g \times [0, 1/2]$ and $E_2 = \Sigma_g \times [1/2, 1]$. Then glue $\Sigma_g \times \{1/2\} \subset E_1$ with $\Sigma_g \times \{1/2\} \subset E_2$ by the identity map except a neighborhood of a_i , namely $a_i \times I \subset \Sigma_g$. Denote the result as E' . By a trivial calculation we get the following presentation:

$$\pi_1(E') = \langle a_1, b_1, a_2, b_2, \dots, a_g, b_g, b'_i \mid \prod_{j=1}^g [a_j, b_j], [a_i, b'_i] \prod_{j \neq i} [a_j, b_j] \rangle$$

Finally we abelianize this presentation after gluing $\Sigma_g \times \{0\} \subset E_1$ with $\Sigma_g \times \{1\} \subset E_2$ using the map ϕ to get $\partial X - \overset{\circ}{\nu}(a_i)$.

$i_*[l'] = a_i$ and $i_*[m'] = b_i - b'_i$ follows from Theorem 4 because a_i intersects b_j only once iff $i = j$.

Second part is obtained similarly.

□

3. APPLICATIONS

First we give an immediate application of the theorems 3 and 4.

Corollary 6. *Let M be a 4-manifold which admits a genus g Lefschetz fibration over D^2 or S^2 . Let n and s be the numbers of nonseparating and separating vanishing cycles in the global monodromy of this fibration, respectively. Then $\sigma(M) \leq n - s$.*

Proof. Suppose that we build up the 4-manifold M from $\Sigma_2 \times D^2$ by attaching 2-handles. By Theorem 4, every time we attach a 2-handle along a separating curve the signature of the resulting 4-manifold will be one less than the signature of the 4-manifold before we attach the 2-handle. Thus Theorem 3 implies the upper bound $n - s$ on the signature. □

Corollary 7. *There is no hyperelliptic Lefschetz fibration (of any genus) over S^2 with only reducible singular fibers. (Here reducible means that the local monodromy corresponding to the singular fiber is a Dehn twist about a separating curve.)*

Proof. Let M be a 4-manifold which admits a Lefschetz fibration of genus g over S^2 with global monodromy $(D(\gamma_1), \dots, D(\gamma_s))$, where $s = \sum_{h=1}^{[\frac{g}{2}]} s_h$ and γ_i is separating for each i , $1 \leq i \leq s$. Then, by the local signature formula,

$$\sigma(M) = \begin{cases} \sum_{h=1}^{[\frac{g}{2}]} \left(\frac{4h(g-h)}{2g+1} - 1 \right) s_h \geq 0 & \text{if } g \geq 3 \\ -s/5 & \text{if } g = 2 \end{cases}$$

But on the other hand $\sigma(M) = -s$ according to Theorem 4. Hence $s = 0$. (This is trivially true for $g = 1$ since any vanishing cycle on a torus is nonseparating.) This proves the desired result since we assume (by definition) that there exists at least one singular fiber in each Lefschetz fibration. □

Next we combine our results with the local signature formula for the hyperelliptic Lefschetz fibrations to give an upper bound for the signatures of these fibrations.

Corollary 8. *Let M be a 4-manifold which admits a hyperelliptic Lefschetz fibration of genus g over S^2 . Let n and s be the numbers of nonseparating and separating vanishing cycles in the global monodromy of this fibration, respectively. Then $\sigma(M) \leq n - s - 4$.*

Proof. We first note that we can improve the inequality

$$\sigma(M) \leq n - s$$

given in Corollary 6 to

$$\sigma(M) \leq n - s - 1$$

for hyperelliptic Lefschetz fibrations as follows:

Suppose that we attach the first 2-handle along a nonseparating curve. We can always assume this because $n \geq 1$ (since we proved in Corollary 7 that $n \neq 0$) and we can cyclically permute the vanishing cycles in the global monodromy of a Lefschetz fibration. Moreover we can easily show that if we start attaching handles along a nonseparating curve then the signature of the resulting 4-manifold (after attaching the very first handle) will be the same as $\sigma(\Sigma_2 \times D^2)$, which is zero.

Next note that $\sigma(M) \leq n - s - 1$ is equivalent to

$$4 \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} h(g-h)s_h \leq (3g+2)n - (2g+1)$$

using the local signature formula.

Assume that g is odd. Endo [E] proves that

$$n + 4 \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} h(2h+1)s_h \equiv 0 \pmod{4(2g+1)}.$$

Hence

$$n = 4c(2g+1) - 4 \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} h(2h+1)s_h$$

for some integer c . Substituting into the inequality above (and dividing by 4) we get

$$\sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} h(g-h)s_h \leq (3g+2)[c(2g+1) - \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} h(2h+1)s_h] - \frac{1}{4}(2g+1).$$

Hence

$$\sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} h(g-h)s_h \leq (3g+2)[c(2g+1) - \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} h(2h+1)s_h] - \frac{1}{4}(2g+2)$$

since $2g+1 \equiv 3 \pmod{4}$.

But this inequality, in turn, implies that

$$4 \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} h(g-h)s_h \leq (3g+2)n - (2g+2)$$

which is equivalent to

$$\sigma(M) \leq n - s - 1 - \frac{1}{2g+1}.$$

Since $\sigma(M)$ is an integer,

$$\sigma(M) \leq n - s - 2.$$

Iterating the same argument, we obtain

$$\sigma(M) \leq n - s - 4.$$

(We use $2(2g+1) \equiv 2 \pmod{4}$ and $3(2g+1) \equiv 1 \pmod{4}$).

Similarly, if g is even, then one can use the corresponding result by Endo:

$$n + 4 \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} h(2h+1)s_h \equiv 0 \pmod{2(2g+1)}.$$

(Note that $2(3g+2)(2g+1) \equiv 0 \pmod{4}$, if g is even.) \square

The following corollary is our main result concerning the *geography* of the hyperelliptic Lefschetz fibrations.

Corollary 9. *For each $g \geq 1$, there exist*

$$k_g \leq 10 - \frac{6g+4}{g^2} \text{ and } l_g \leq 2g-10 + \frac{4g+4}{g^2}$$

such that if a 4-manifold admits a hyperelliptic Lefschetz fibration of genus g over S^2 then its signature σ and its Euler characteristic e satisfy the inequality $c_1^2 \leq k_g\chi + l_g$, where $c_1^2 = 3\sigma + 2e$ and $\chi = \frac{1}{4}(\sigma + e)$.

Proof. Let M be a 4-manifold which admits a hyperelliptic Lefschetz fibration of genus g over S^2 .

Let n and $s = \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} s_h$ be the numbers of nonseparating and separating vanishing cycles in the global monodromy of this fibration, respectively. Then

$$\sigma(M) = -\frac{g+1}{2g+1}n + \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} \left(\frac{4h(g-h)}{2g+1} - 1 \right) s_h$$

and

$$e(M) = n + \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} s_h - 4(g-1).$$

Thus

$$c_1^2(M) = 3\sigma(M) + 2e(M) = \frac{1}{2g+1}[(g-1)n - 8(g-1)(2g+1) + \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} (12h(g-h) - 2g-1)s_h]$$

and

$$\chi(M) = \frac{\sigma(M) + e(M)}{4} = \frac{1}{4(2g+1)}[gn - 4(g-1)(2g+1) + \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} 4h(g-h)s_h].$$

We want to find k_g and l_g such that $c_1^2(M) \leq k_g\chi(M) + l_g$. Using the equalities above we get,

$$\sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} [(12 - k_g)h(g-h) - 2g-1]s_h \leq [\frac{k_g}{4}g - (g-1)]n + m_g,$$

where $m_g = -(g-1)(2g+1)k_g + (2g+1)l_g + 8(g-1)(2g+1)$. Now we note that

$$\sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} [(12 - k_g)h(g-h) - 2g - 1]s_h = \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} h(g-h)(12 - k_g - \frac{2g+1}{h(g-h)})s_h.$$

If g is even, then

$$\sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} h(g-h)(12 - k_g - \frac{2g+1}{h(g-h)})s_h \leq (12 - k_g - \frac{2g+1}{g^2/4}) \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} h(g-h)s_h.$$

By Corollary 8,

$$(12 - k_g - \frac{2g+1}{g^2/4}) \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} h(g-h)s_h \leq \frac{1}{4}[(3g+2)n - 4(2g+1)](12 - k_g - \frac{2g+1}{g^2/4}).$$

Hence it suffices to find k_g and l_g such that

$$\frac{1}{4}[(3g+2)n - 4(2g+1)](12 - k_g - \frac{2g+1}{g^2/4}) \leq [\frac{k_g}{4}g - (g-1)]n + m_g.$$

First consider the inequality

$$\frac{1}{4}(3g+2)(12 - k_g - \frac{2g+1}{g^2/4}) \leq \frac{k_g}{4}g - (g-1).$$

Solving for k_g we get,

$$k_g \geq 10 - \frac{6g+4}{g^2}.$$

Hence it suffices to find l_g such that

$$-(2g+1)(12 - k_g - \frac{2g+1}{g^2/4}) \leq m_g = -(g-1)(2g+1)k_g + (2g+1)l_g + 8(g-1)(2g+1).$$

Solving for l_g we get,

$$l_g \geq (k_g - 8)(g-1) - 12 + k_g + \frac{8g+4}{g^2}.$$

Therefore, if g is even, then

$$k_g = 10 - \frac{6g+4}{g^2}$$

and

$$l_g = 2g - 10 + \frac{4g+4}{g^2}$$

will satify the requirements of the statement in the theorem.

Similarly, if g is odd and $g \geq 3$, then one can calculate that

$$k_g = 10 - \frac{6g+4}{g^2-1}$$

and

$$l_g = 2g - 10 + \frac{8}{2g+1} + \frac{6}{(g^2-1)(2g+1)}$$

will suffice.

Moreover, we can choose $k_1 = l_1 = 0$ since $c_1^2 = 0$ for all genus one Lefschetz fibrations over S^2 .

It is trivial to verify that $k_g \leq 10 - \frac{6g+4}{g^2}$ and $l_g \leq 2g - 10 + \frac{4g+4}{g^2}$ for all $g \geq 1$.

□

Remarks . 1. In [S2], Stipsicz proves that $0 \leq c_1^2 \leq 10\chi$ for the (relatively minimal) genus $g(\geq 1)$ Lefschetz fibrations over closed surfaces of nonzero genus. However, his result does not extend to cover the fibrations over S^2 . He also points out that $c_1^2 \leq 10\chi + 2g - 1$ holds for any Lefschetz fibration.

2. In particular, Corollary 9 yields $k_2 = 6$, $l_2 = -3$, $k_3 = 7.25$, $l_3 = -2.75$, $k_4 = 8.25$ and $l_4 = -0.75$.

3. Combining our results for genus two and genus three Lefschetz fibrations, we have shown the following: The signature of a smooth 4-manifold which admits a hyperelliptic Lefschetz fibration of genus $g \leq 3$ over S^2 is negative.

Hence we can formulate the following natural question: What is the minimal g such that a smooth 4-manifold with positive signature admits a (hyperelliptic) Lefschetz fibration of genus g over S^2 ?

4. If one can improve the inequality

$$\sigma(M) \leq n - s - 4$$

in Corollary 8 to

$$\sigma(M) \leq n - s - 4(g - 1)$$

then one can prove that $c_1^2 \leq 10\chi$ for all hyperelliptic Lefschetz fibrations over S^2 .

4. EXAMPLES

4.1. GENUS 1.

To illustrate how one can develop an algorithm using our main theorems to calculate the signatures of smooth Lefschetz fibrations, we will give the details of our computation to obtain the well-known result $\sigma(E(1)) = -8$ for the elliptic surface $E(1)$.

The global monodromy of $E(1)$ is given by the sequence $(\alpha, \beta)^6$ of 12 Dehn twists where $\alpha = D(a)$ and $\beta = D(b)$ denote the positive Dehn twists about the curves a and b , respectively (Figure 3).

To build up $E(1)$, we start with a copy of $T^2 \times D^2$ and glue 2-handles along the vanishing cycles a and b in an alternating fashion. (We will use the letters a and b also to denote the homology classes of the curves a and b , respectively.)

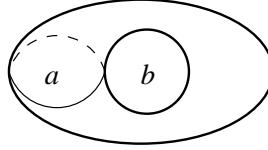


FIGURE 3.

Let ϕ denote the monodromy of the boundary of the 4-manifold before we attach a 2-handle.

Let $C = \text{Ker}(i_*)$ in the $\{[m'], [l']\}$ plane, when we attach a 2-handle (cf. Theorem 4). Below we use Proposition 5 to compute $i_*[l']$ and $i_*[m']$.

$\phi = \text{identity}$, attach the first handle along a ,

$i_*[l'] = a$ and $i_*[m'] = b - b' = 0$.

$C = \langle [m'] \rangle$, $\sigma(id, a) = 0$

$\phi = \alpha$, attach the second handle along b ,

$i_*[l'] = b$ and $i_*[m'] = a' - a = 0$.

$C = \langle [m'] \rangle$, $\sigma(\alpha, b) = 0$

$\phi = \beta\alpha$, attach the third handle along a ,

$i_*[l'] = a$ and $i_*[m'] = b - b' = -a$.

$C = \langle [m'] + [l'] \rangle$, $\sigma(\beta\alpha, a) = -1$

$\phi = \alpha\beta\alpha$, attach the fourth handle along b ,

$i_*[l'] = b$ and $i_*[m'] = a' - a = -2b$.

$C = \langle [m'] + 2[l'] \rangle$, $\sigma(\alpha\beta\alpha, b) = -1$

$\phi = \beta\alpha\beta\alpha$, attach the fifth handle along a ,

$i_*[l'] = a$ and $i_*[m'] = b - b' = 3b$.

$C = \langle [m'] + 3[l'] \rangle$, $\sigma(\beta\alpha\beta\alpha, a) = -1$

$\phi = \alpha\beta\alpha\beta\alpha$, attach the sixth handle along b ,

$i_*[l'] = b = 0$ and $i_*[m'] = a' - a = -2a$.

$C = \langle [l'] \rangle$, $\sigma(\alpha\beta\alpha\beta\alpha, b) = -1$

$\phi = \beta\alpha\beta\alpha\beta\alpha$, attach the seventh handle along a ,

$i_*[l'] = a = 0$ and $i_*[m'] = b - b' = 2b$.

$C = \langle [l'] \rangle$, $\sigma(\beta\alpha\beta\alpha\beta\alpha, a) = -1$

$\phi = \alpha\beta\alpha\beta\alpha\beta\alpha$, attach the eighth handle along b ,

$i_*[l'] = b$ and $i_*[m'] = a' - a = 4b$.

$C = \langle -[m'] + 4[l'] \rangle$, $\sigma(\alpha\beta\alpha\beta\alpha\beta\alpha, b) = -1$

$\phi = \beta\alpha\beta\alpha\beta\alpha\beta\alpha$, attach the ninth handle along a ,

$i_*[l'] = a$ and $i_*[m'] = b - b' = 3a$.

$C = < -[m'] + 3[l'] >$, $\sigma(\beta\alpha\beta\alpha\beta\alpha\beta\alpha, a) = -1$

$\phi = \alpha\beta\alpha\beta\alpha\beta\alpha\beta\alpha$, attach the tenth handle along b ,

$i_*[l'] = b$ and $i_*[m'] = a' - a = 2b$.

$C = < -[m'] + 2[l'] >$, $\sigma(\alpha\beta\alpha\beta\alpha\beta\alpha\beta\alpha, b) = -1$

$\phi = \beta\alpha\beta\alpha\beta\alpha\beta\alpha\beta\alpha$, attach the eleventh handle along a ,

$i_*[l'] = a$ and $i_*[m'] = b - b' = a$.

$C = < -[m'] + [l'] >$, $\sigma(\beta\alpha\beta\alpha\beta\alpha\beta\alpha\beta\alpha, a) = 0$

$\phi = \alpha\beta\alpha\beta\alpha\beta\alpha\beta\alpha\beta\alpha$, attach the twelfth handle along b ,

$i_*[l'] = b$ and $i_*[m'] = a' - a = b$.

$C = < -[m'] + [l'] >$, $\sigma(\alpha\beta\alpha\beta\alpha\beta\alpha\beta\alpha\beta\alpha, b) = 0$

Therefore by Theorem 3

$$\sigma(E(1)) = \sigma(id, a) + \sigma(\alpha, b) + \sigma(\beta\alpha, a) + \cdots + \sigma(\alpha\beta\alpha\beta\alpha\beta\alpha\beta\alpha\beta\alpha, b) = -8.$$

4.2. GENUS 2.

We developed a Mathematica program to compute the signature of a 4-manifold which admits a Lefschetz fibration over D^2 or S^2 whose global monodromy is given by any finite sequence of positive Dehn twists $D(c_1), D(c_2), \dots, D(c_5)$, where c_1, \dots, c_5 are the curves indicated in Figure 4.

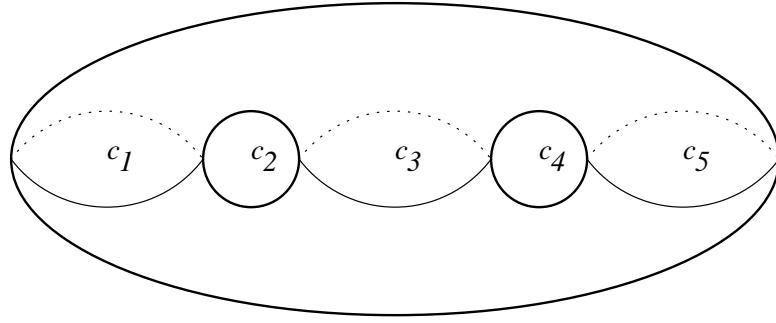


FIGURE 4.

We computed the signatures of the following genus two Lefschetz fibrations of some closed 4-manifolds, using our Mathematica program (which is available upon request).

Let ζ_i denote $D(c_i)$, $1 \leq i \leq 5$.

$$\sigma((\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_5, \zeta_4, \zeta_3, \zeta_2, \zeta_1)^2) = -12,$$

$$\sigma((\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5)^6) = -18,$$

$$\begin{aligned}\sigma((\zeta_5, \zeta_1, \zeta_4, \zeta_2, \zeta_3, \zeta_4, \zeta_2, \zeta_5, \zeta_1, \zeta_4, \zeta_2, \zeta_3, \zeta_4, \zeta_2, \zeta_3)^2) &= -18, \\ \sigma((\zeta_1, \zeta_2, \zeta_3, \zeta_4)^{10}) &= -24.\end{aligned}$$

Remarks . 1. One can indeed check these numbers using Matsumoto's local signature formula.

2. All of the examples above are fibrations with even number of singular fibers. We want to point out that there are also examples with odd number of singular fibers in a Lefschetz fibration. Just take Matsumoto's example with 8 singular fibers and replace one of the separating twists with the sequence of 12 nonseparating twists $(\zeta_1, \zeta_2)^6$ so that the resulting fibration will have 19 singular fibers. If we replace the remaining separating twist with another sequence of 12 nonseparating twists then we get a fibration with 30 singular fibers. It is a question whether the resulting manifold is diffeomorphic to $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5)^6$ or not.

We can generalize this to any genus two Lefschetz fibration over S^2 :

Proposition 10. *Let M be a 4-manifold which admits a genus two Lefschetz fibration over S^2 . Let M' be the resulting manifold obtained by replacing a separating vanishing cycle in the global monodromy of this fibration by the sequence $(\zeta_1, \zeta_2)^6$. Then*

$$\sigma(M') = \sigma(M) - 7$$

and

$$e(M') = e(M) + 11.$$

Proof. We will give two different proofs, one using Matsumoto's local signature formula, the other using the technique we developed in this paper. Let n and s be the numbers of nonseparating and separating vanishing cycles for M , respectively. Replacing one of the separating cycles by a product of 12 nonseparating cycles we get $(n + 12)$ nonseparating and $(s - 1)$ separating cycles for M' . Thus

$$\sigma(M') = (-3/5)(n + 12) + (-1/5)(s - 1) = (-3/5)n + (-1/5)s - 7 = \sigma(M) - 7.$$

Also using part 2 in Theorem 4 when we remove a separating twist the signature will increase by 1. Gluing $(\zeta_1, \zeta_2)^6$ we add -8 to the signature because the computation in genus two will be the same as in genus one ($\sigma((\alpha, \beta)^6) = -8$) using our technique. Hence we get the same result.

Moreover $e(M') = e(M) + 11$ since we replace a singular fiber with 12 singular fibers.

□

We also computed the signatures of the following genus two Lefschetz fibrations over D^2 :

$$\sigma((\zeta_1, \zeta_3, \zeta_5, \zeta_2, \zeta_4, \zeta_2, \zeta_1, \zeta_1, \zeta_3, \zeta_5, \zeta_3, \zeta_5, \zeta_2, \zeta_4, \zeta_5)) = -10.$$

$$\sigma((\zeta_1, \zeta_3, \zeta_5, \zeta_2, \zeta_4, \zeta_2, \zeta_1, \zeta_1, \zeta_3, \zeta_5, \zeta_3, \zeta_5, \zeta_2, \zeta_4, \zeta_5)^2) = -20.$$

$$\sigma((\zeta_1, \zeta_3, \zeta_5, \zeta_2, \zeta_4, \zeta_2, \zeta_1, \zeta_1, \zeta_3, \zeta_5, \zeta_3, \zeta_5, \zeta_2, \zeta_4, \zeta_5)^3) = -30.$$

Remark . We can not use the local signature formula in order to check these results because it only works when the base space is S^2 (or any closed surface).

4.3. GENUS 3.

We computed the signatures of the following genus three Lefschetz fibrations over S^2 , using our Mathematica program. (The program is available upon request.)

$$\sigma((\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6)^{14}) = -48$$

$$\sigma((\eta_8, \eta_9, \eta_4, \eta_3, \eta_3, \eta_2, \eta_1, \eta_5, \eta_4, \eta_3, \eta_2, \eta_6, \eta_5, \eta_4, \eta_3, (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6)^{10})) = -42.$$

(This is a word in \mathcal{M}_3 given in [F]).

Here $\eta_1, \eta_2, \dots, \eta_9$ denote the positive Dehn twists about the curves d_1, d_2, \dots, d_9 indicated as in Figure 5.

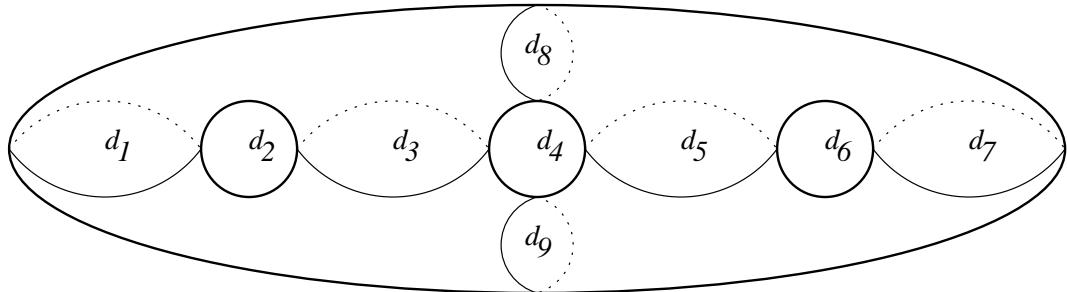


FIGURE 5.

Remark . The latter of these fibrations is not hyperelliptic since otherwise the local signature formula [E] would yield $\sigma = 74(-4/7)$ which is not an integer!

We also computed the signatures of the following genus three Lefschetz fibrations over D^2 :

$$\sigma((\eta_1, \eta_3, \eta_5, \eta_7, \eta_2, \eta_4, \eta_6)) = -1$$

$$\sigma((\eta_1, \eta_3, \eta_5, \eta_7, \eta_2, \eta_4, \eta_6)^2) = -8$$

$$\sigma((\eta_1, \eta_3, \eta_5, \eta_7, \eta_2, \eta_4, \eta_6)^3) = -11$$

$$\sigma((\eta_1, \eta_3, \eta_5, \eta_7, \eta_2, \eta_4, \eta_6)^4) = -16$$

5. FINAL REMARK

Given a product of positive Dehn twists in the mapping class group of a genus g surface, we can construct a symplectic 4-manifold which admits a Lefschetz fibration over D^2 , as we have studied in this paper. A natural generalization is to allow negative Dehn twists also. These fibrations are called *achiral* Lefschetz fibrations. Our technique clearly extends to compute the signatures of these fibrations.

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